Risk-Reward Trade-off

Scenario Analysis

Scenario I You purchase $1000 worth of a certain stock and hold it for one year at which time you sell it.

Market Forecast	Prob.	Payoff	Profit	Rate of Return
Bull	.5	$1500	$500	50%
Bear	.5	$800	-$200	-20%

Note: Rate of Return $= \dfrac{\text{Profit}}{\text{Initial Investment}} \times 100$

In the above situation the *expected* or *average rate of return* is given by

$$\bar{r} = .5(50) + .5(-20) = 15\%$$

$\bar{r}$ is a measurement of expected *reward*.

Scenario II Still a \$1000 investment

Market Forecast	Prob.	Payoff	Profit	Rate of Return
Bull	.6	\$1150	\$150	15%
Bear	.4	\$1150	\$150	15%

Expected or average rate of return is given by

$$\overline{r} = .6(15) + .4(15) = 15\%$$

Scenario III: Once again a $1000 investment

Market Forecast	Prob.	Payoff	Profit	Rate of Return
Bull	.1	7900	6900	690%
Bear	.9	400	-600	-60%

Expected or average rate of return is given by

$$\overline{r} = .1(690) + .9(-60) = 15\%$$

once again.

Discussion:

1. Which scenario do *you* prefer?

2. Are you risk averse or a risk lover or somewhere in the middle?

3. Rank the three scenarios in terms of risk.

- Scenario I:

Market Forecast	Prob.	Payoff	Profit	Rate of Return
Bull	.5	$1500	$500	50%
Bear	.5	$800	-$200	-20%

- Scenario II:

Market Forecast	Prob.	Payoff	Profit	Rate of Return
Bull	.6	$1150	$150	15%
Bear	.4	$1150	$150	15%

- Scenario III:

Market Forecast	Prob.	Payoff	Profit	Rate of Return
Bull	.1	7900	6900	690%
Bear	.9	400	-600	-60%

Note: In Scenario II the return is *riskless*, would you agree?

Which appears riskier – Scenario I or Scenario III?

Measurement of Risk

An accepted *measure of risk* is the Standard Deviation σ of Returns. It is computed as follows:

Scenario I

$$\sigma_I = \sqrt{.5(.50 - .15)^2 + .5(-.20 - .15)^2}$$

$$= .35$$

Scenario II

$$\sigma_{II} = \sqrt{.6(.15 - .15)^2 + .4(.15 - .15)^2}$$

$$= 0$$

Scenario III

$$\sigma_{III} = \sqrt{.1(6.90 - .15)^2 + .9(-.60 - .15)^2}$$

$$= 2.25$$

Summarizing

Scenario I

$$\sigma_I = \sqrt{.5(.50 - .15)^2 + .5(-.20 - .15)^2} = .35$$

Scenario II

$$\sigma_{II} = \sqrt{.6(.15 - .15)^2 + .4(.15 - .15)^2} = 0$$

Scenario III

$$\sigma_{III} = \sqrt{.1(6.90 - .15)^2 + .9(-.60 - .15)^2} = 2.25$$

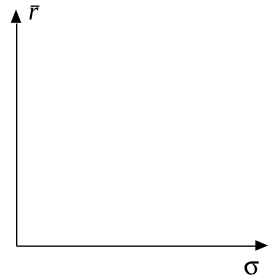
Note: In Scenario II, σ_{II}, the measure of risk, is 0.

Also

$$\sigma_I < \sigma_{III}$$

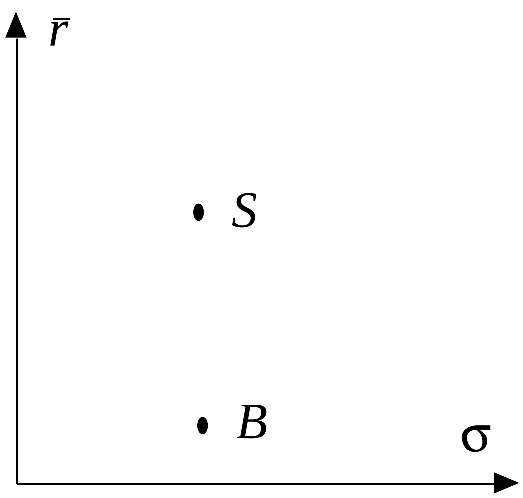
indicating that σ_{III} carries the most risk of the three scenarios.

Risk-Reward Plots for Securities S and B



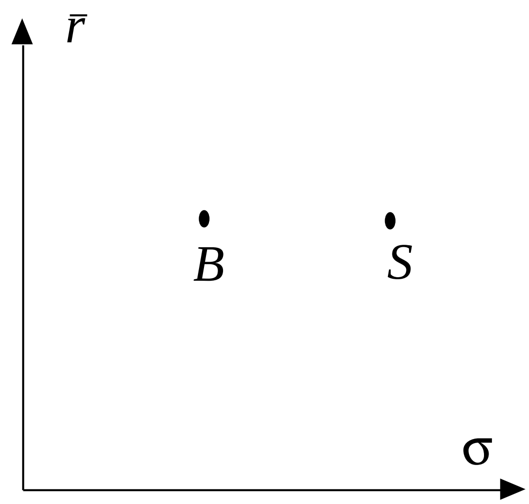
Risk-Reward space

Preferences:



$\sigma_S = \sigma_B$

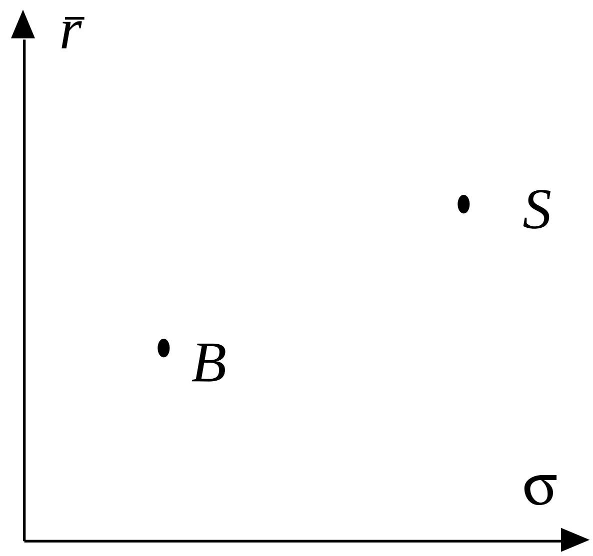
$\bar{r}_S > \bar{r}_B$



$\bar{r}_B = \bar{r}_S$

$\sigma_B < \sigma_S$

Which would you prefer?



$\bar{r}_B < \bar{r}_S$ $\sigma_B < \sigma_S$

Which would you prefer?

Here, one can't say which is preferred without further assumptions.

Utility Functions:

Utility functions help us specify preferences – the concept of utility function comes from economics.

For our risk-reward theory a commonly used utility function for an individual's preference is

$$U(S) = \bar{r}_S - \frac{1}{2} A \sigma_S^2$$

This utility function represents a trade-off between return (reward) and risk.

The positive constant A is a measure of the individual's *aversion to risk*.

In our examples, $0 \le A \le 4$.

$A = 4$ means *risk-averse*, $A = 0$ means *risk-lover*.

Example: $A = 2$

If a stock S_1 has a standard deviation σ_{s_1} of 20% and an expected return $\overline{r}_{S_1}$ of 16% then

$$U(S_1) = .16 - \frac{1}{2}(2)(.20)^2$$

$$= .12$$

If a second stock has of a standard deviation of 30% but an expected return 22% then

$$U(S_2) = .22 - \frac{1}{2}(2)(.30)^2$$

$$= .13$$

and an investor with risk aversion constant $A = 2$ would prefer S_2 over S_1.

In the sketch, click the *Utility On* button. A moveable point Q appears with its utility. You can adjust A.

However if $A = 4$ our investor is now more *risk-averse*. Then

$$U(S_1) = .16 - \frac{1}{2}(4)(.20)^2$$

$$= .08$$

while

$$U(S_2) = .22 - \frac{1}{2}(4)(.30)^2$$

$$= .04$$

so our new more risk-averse investor would prefer S_1 over S_2.

Note: If $U(S_1) = U(S_2)$ our investor will be *indifferent* to the choice between S_1 and S_2 – the two choices look equally favorable to him/her.

Indifference Curves:

Consider all those assets S "equivalent" in terms of Utility to a given asset S_*

$$\text{i.e. } U(S) = U(S_*)$$

$$\text{or} \quad \overline{r}_S - \tfrac{1}{2}A\sigma_S^2 = U(S_*)$$

Now letting $U(S_*) = c$,

We have

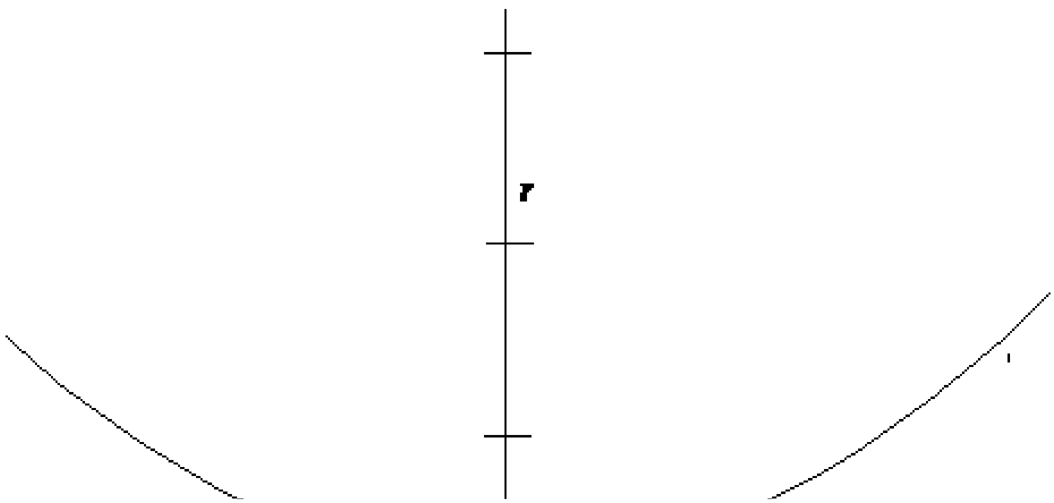
$$\overline{r}_S = \frac{1}{2}A\sigma_S^2 + c$$

If we now plot all pairs $(\sigma, \overline{r})$ that satisfy

$$\overline{r} = \frac{1}{2}A\sigma^2 + c$$

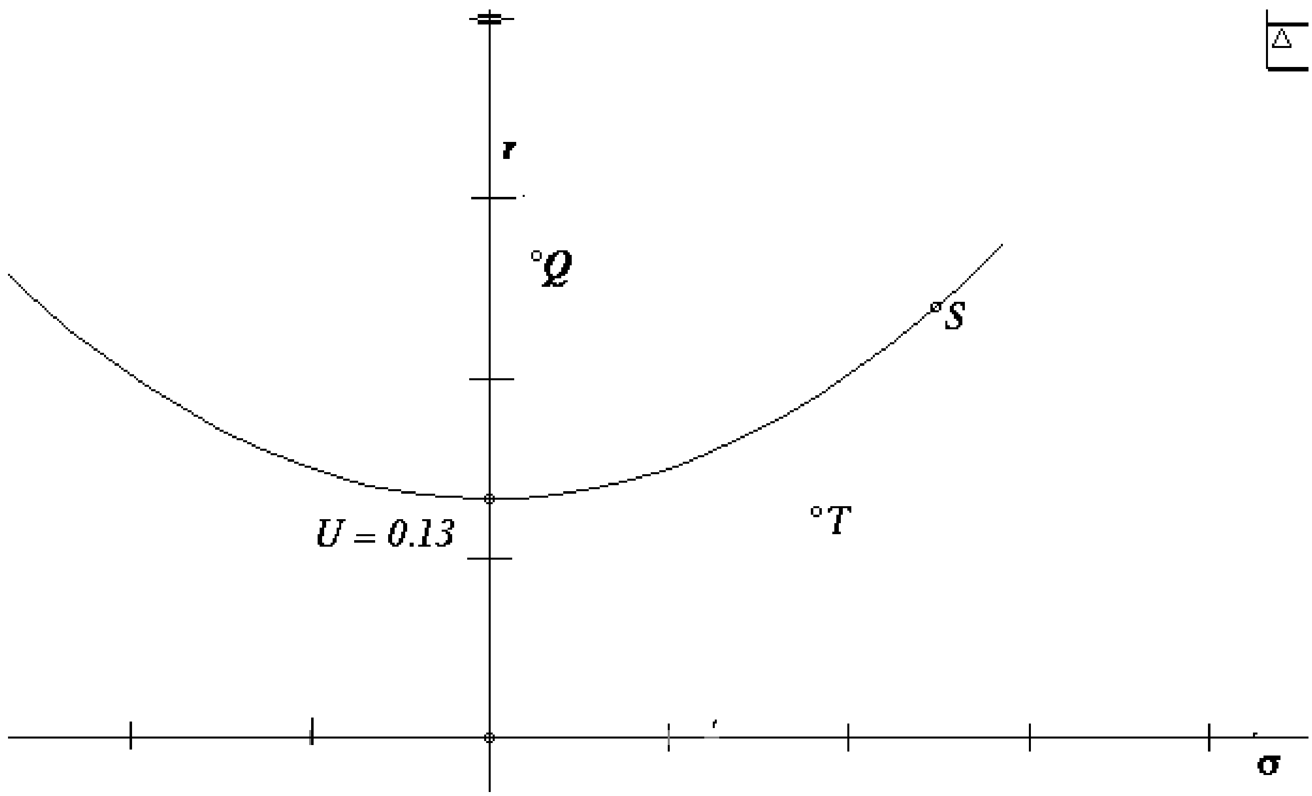
(think of parabolas $y = \frac{1}{2}Ax^2 + c$)

we get an *indifference curve* corresponding to the utility $c = U(S_*)$



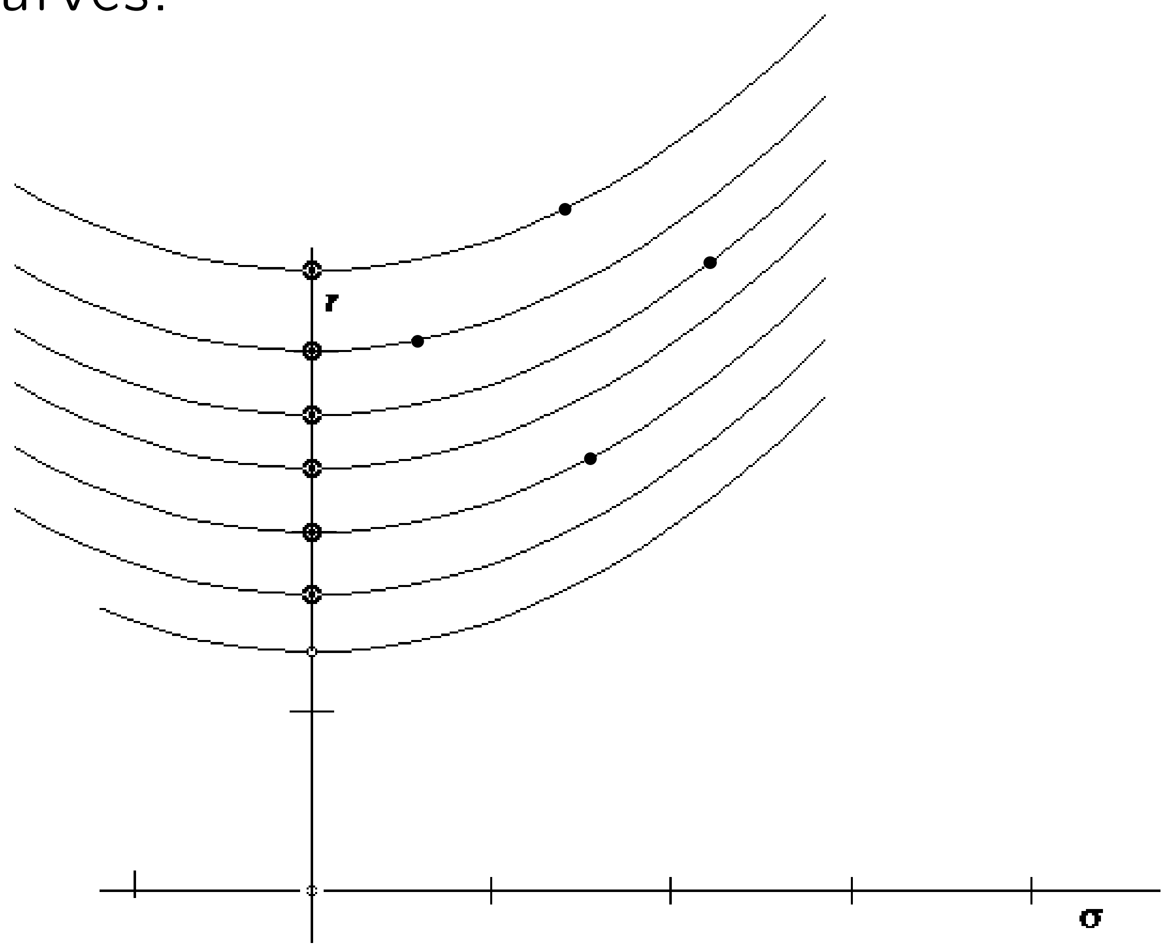
$$\overline{r}_S = \frac{1}{2} \cdot 3.47 \cdot \sigma_S^2 + .13$$

Here $A = 3.47$ and $c = U = .13$. The parabola is a *contour line* for the utility function.



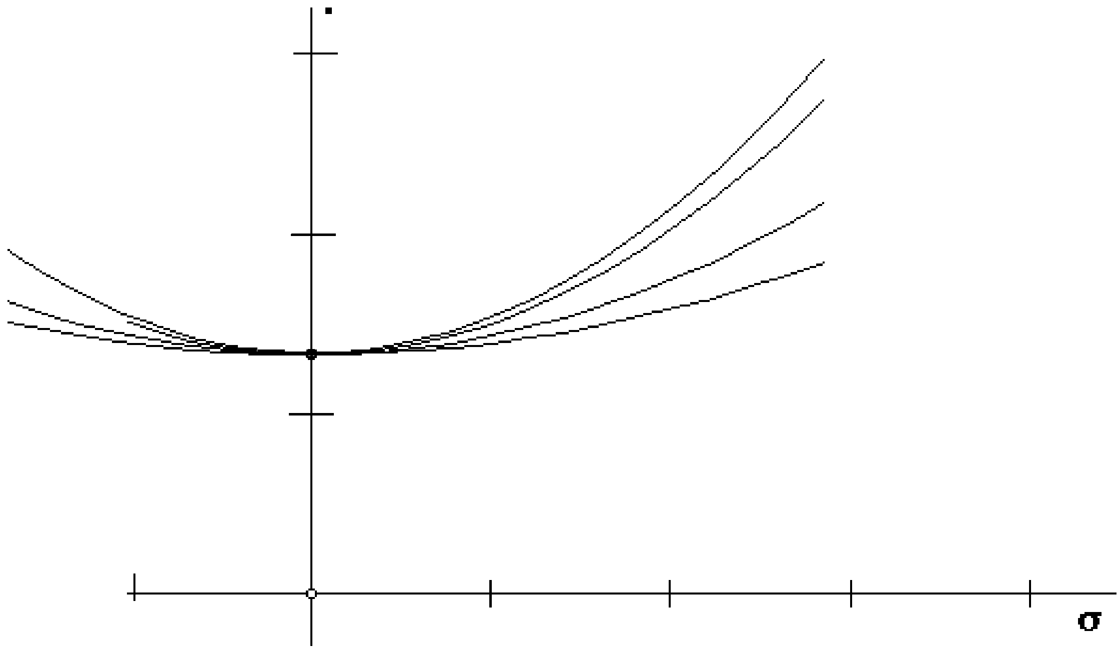
Which would you prefer? Q, S, or T?

As we vary c, we get a *family* of indifference curves.



c (the utility) varies and A is constant

Keeping U constant and letting A vary, we get another family:



c (the utility) is constant and A varies

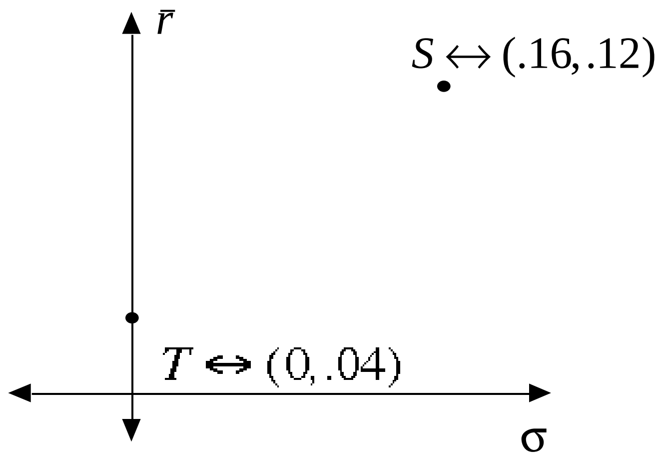
How can you tell a risk lover from a risk avoider by the shape of the parabola?

Recall: An investor is indifferent to the choice between any two investments on a *given indifference curve*.

In the sketch, check out the *Contour Line* button. Adjust A.

Construction of a Portfolio:
Mixing one risky asset S (Index mutual fund, say) and a risk-free money market account (or CD T),

Question: How to allocate \$10,000 between index fund S and money market account T:



Risky S, Safe T

Suppose $0 \leq \alpha \leq 1$, and you invest a fraction "α" of your money in S; then you'll invest $1 - \alpha$ of it in T.

(Think: $\frac{1}{3}$ in S and $\frac{2}{3}$ in T)

What point P on the risk-reward plane corresponds to a fraction α in S and $(1 - \alpha)$ in T?

Symbolically,

$$``P(\alpha) = \alpha S + (1-\alpha)T''$$

It turns out that in this case,

$$\bar{r}_P = \alpha\bar{r}_S + (1-\alpha)\bar{r}_T$$

and because $\sigma_T = 0$

$$\sigma_P = \alpha\sigma_S$$

So,

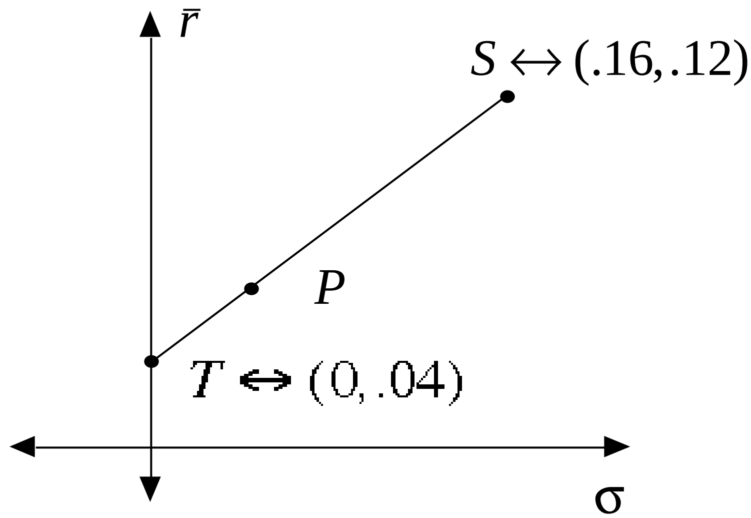
$$P \leftrightarrow (\alpha\sigma_S, \alpha\bar{r}_S + (1-\alpha)\bar{r}_T)$$

$$= (\alpha\sigma_S + (1-\alpha)\sigma_T, \alpha\bar{r}_S + (1-\alpha)\bar{r}_T)$$

$$= \alpha(\sigma_S, \bar{r}_S) + (1-\alpha)(\sigma_T, \bar{r}_T)$$

$$= \alpha S + (1-\alpha)T$$

So, in this case, our symbolic equation is an actual equation. As α varies from 0 to 1, P travels along the line segment $\overline{TS}$:



P is a point on $\overline{TS}$

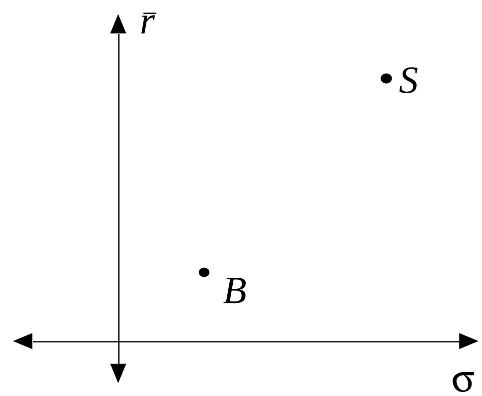
$$P = \alpha S + (1 - \alpha)T$$

When $\alpha = 0$, all your money is in T. When $\alpha = 1$, all your money is in S.

In the sketch, check out the *Path On* button. Adjust α.

The more general case

Suppose, as before, you split your money up among two investments:



S and B both have some risk

As before, you construct a portfolio P by investing a fraction α in S and $(1-\alpha)$ in B:

$$\text{“}P(\alpha) = \alpha S + (1-\alpha)B\text{”}$$

Now there is a correlation coefficient, ρ, between S and B, so the $(\sigma_P, \overline{r}_P)$ coordinates of P in the risk-reward plane are no longer linear in S and B.

Well, the second coordinate *is* linear:

$$\bar{r}_P = \alpha \bar{r}_S + (1 - \alpha)\bar{r}_B$$

But the *first* coordinate of P is more complicated:

$$\sigma_P = \sqrt{\alpha^2 \sigma_S^2 + 2\,\rho\,\alpha(1-\alpha)\sigma_S \sigma_B + (1-\alpha)^2 \sigma_B^2}$$

As α goes from 0 to 1, P traces out a *path* from B to S that is parameterized by α. Only in special cases is it a straight line segment.

(Look at the algebra and predict when the path is a straight line.)

In the sketch, move B off the vertical axis. Click the *Indifference Curve* and *Contour On* buttons on. Adjust ρ. For given values of parameters, how can you experimentally maximize your reward? Optimize your portfolio? What happens if $\rho = -1$?